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# Prolongation structures of a higher-order nonlinear Schrödinger equation

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**Abstract.** A higher-order Schrödinger equation containing parameters, which is used to describe pulse propagation in optical fibres, is shown to admit an infinite-dimensional prolongation structure for *exactly* four combinations of the parameters, besides the classical NLS equation. For each of these cases, the structure of the resulting prolongation algebra is determined explicitly. For the first three cases the prolongation algebra is essentially a sub-algebra of $A_1^{(1)}$, the fourth case turns out to be a sub-algebra of the *twisted* Kac–Moody algebra $A_2^{(2)}$. Using vector-field representations, related systems of differential equations for the (pseudo-) potential functions are given for each of the cases . The cases found here correspond exactly to the derived NLS equations I and II, the Hirota equation and the equation recently considered by Sasa and Satsuma. The result of this paper strongly indicates that the considered higher-order NLS equation is completely integrable for *precisely* these four cases.

## 1. Introduction

The nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}\psi_t + \tfrac{1}{2}\psi_{xx} + |\psi|^2\psi = 0 \tag{1}$$

describes the envelope of slowly varying waves in a large number of applications. In particular, it governs light pulses in optical fibres. Hasegawa wrote an extensive introduction on this subject [2]. Starting from the Maxwell equations, Kodama and Hasegawa proposed the following higher-order NLS equation [2, 5].

$$\mathrm{i}(\psi_t + \Gamma\psi) + \frac{1}{2}\psi_{xx} + |\psi|^2\psi + \varepsilon\mathrm{i}\left[\beta_1\psi_{xxx} + \beta_2\frac{\partial}{\partial x}(|\psi|^2\psi) + \beta_3\psi\frac{\partial}{\partial x}|\psi|^2\right] = 0\,. \tag{2}$$

The parameter $\Gamma$, which can be seen to be a damping coefficient, is supposed to be real. Notice that the notation is changed with respect to [2, 4, 5].

Until recently, only three sets of parameters, besides the original NLS equation (1), were known to lead to soliton-like behaviour (not counting the original NLS equation, with $\epsilon = \Gamma = 0$). Lately, Sasa and Satsuma found a fourth case [9].

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In this paper we will show that the four cases mentioned above are just *all* the cases admitting infinite-dimensional prolongation structures. This indicates that *all* the equations of type (2), that are completely integrable, have been found.

After a short introduction to the Wahlquist and Estabrook prolongation method, we shall give the defining relations and determine the explicit structure of the resulting Lie algebra for all cases. In all cases they turn out to be sub-algebras of the Kac–Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$. Using vector field representations we construct related differential equations which, for instance, may be used to find Bäcklund transforms.

Since the equations become rather involved, the use of computer algebra is almost imperative. For the calculations, a package for working with vector fields [1] and a package for working with Lie algebras [8] were used.

## 2. Prolongation structures

Wahlquist and Estabrook found a method of systematically deriving conservation laws, by means of prolongation structures. They applied it to the Korteweg–de Vries equation [10] and to the NLS equation [11].

Following Vinogradov and Krasil'shchik the prolongation method can be described in terms of vector fields as follows [6]. To a given evolution equation

$$u_t = f[u, u_x, \ldots]$$

we can associate so-called total differential operators

$$D_x = \partial_x + \sum_{i=0}^{\infty} u_{i+1}\partial_{u_i}$$

$$D_t = \partial_t + \sum_{i=0}^{\infty} u_{i,t}\partial_{u_i}$$

where $u_1 = u_x$, $u_2 = u_{xx}$, etc. and $u_{i,t} = D_x^i u_t$. One verifies that $[D_x, D_t] = 0$.

The prolongation method now consists of extending the space of dependent variables $U$ by a finite-dimensional manifold $Y$ with local coordinates $(y_1, \cdots, y_n)$ and at the same time extending $D_x$ and $D_t$ to $U \times Y$. If we put

$$\begin{aligned} \tilde{D}_x &= D_x + X \\ \tilde{D}_t &= D_t + T \end{aligned} \tag{3}$$

where $X = \sum X_i \partial_{y_i}$ and $T = \sum T_i \partial_{y_i}$ are vector fields on $Y$, and $X_i$, $T_i$ are functions on $U \times Y$, we require the formal integrability condition

$$[\tilde{D}_x, \tilde{D}_t] = D_x T - D_t X + [X, T] = 0\,. \tag{4}$$

One verifies that $X_i = \tilde{D}_x y_i \equiv y_{i,x}$ and $T_i = \tilde{T}_x y_i \equiv y_{i,t}$ and that the integrability condition is equivalent to requiring that $y_{i,xt} = y_{i,tx}$.

Applying the Wahlquist and Estabrook prolongation method is equivalent to taking $X = X(\psi, \overline{\psi})$, where $X$ is a vector-field-valued function on $Y$. For equation (2) condition (4) gives rise to an overdetermined system of differential equations which can be solved to give

$$X = x_1 + \psi z_1 + \overline{\psi}\overline{z}_1 + \psi^2 z_2 + \overline{\psi}^2 \overline{z}_2 + \psi\overline{\psi} x_2 \tag{5}$$

where the $x_i$, $y_i$ and $z_i$ are vector fields on a yet unknown manifold $Y$. The term in $T$ without $\psi$ dependence is denoted by $x_3$. As far as their $Y$ dependency is concerned, the higher-order terms in $T$ are elements of the free Lie algebra generated by $x_1$, $x_2$, $x_3$, $z_1$, $\overline{z}_1$, $z_2$ and $\overline{z}_2$. We follow the use of Wahlquist and Estabrook [11] in denoting real algebra elements by the letter $x$, imaginary elements by the letter $y$, and complex elements by the letter $z$.

When all commutators with $z_k$ are known, the commutators with $\overline{z}_k$ are known too, since $[x_i, \overline{z}_k] = \overline{[x_i, z_k]}$, $[y_i, \overline{z}_k] = -\overline{[y_i, z_k]}$, $[z_i, \overline{z}_k] = \overline{[\overline{z}_i, z_k]}$ and $[\overline{z}_i, \overline{z}_k] = \overline{[z_i, z_k]}$. Furthermore, all the $x_i$ and $y_j$ will turn out to commute among themselves, so only the commutators with the $z_k$ need to be given to define the Lie algebra. Whenever a relation with a $z_k$ is given, the complex-conjugated relation is implied as well. For instance, $[z_1, z_3] = 0$ implies $[\overline{z}_1, \overline{z}_3] = 0$ and $[y_1, z_1] + z_1 = 0$ implies $-[y_1, \overline{z}_1] + \overline{z}_1 = 0$.

From the prolongation structure, conservation laws can be deduced. In particular, when the prolongation structure is infinite-dimensional, in some cases an infinite number of conservation laws can be constructed, proving the complete integrability of the equation. This indicates that equations admitting infinite-dimensional prolongation structures are completely integrable.

*Theorem 1.* Besides the classical NLS equation (1), equation (2) has an infinite-dimensional prolongation structure in exactly the following four cases.

(i) $\varepsilon(\beta_1, \beta_2, \beta_3) = \delta(0, 1, 0)$
(ii) $\varepsilon(\beta_1, \beta_2, \beta_3) = \delta(0, 1, -1)$
(iii) $\varepsilon(\beta_1, \beta_2, \beta_3) = \delta(1, 6, -6)$
(iv) $\varepsilon(\beta_1, \beta_2, \beta_3) = \delta(1, 6, -3)$

In all cases, $\delta$ must be real, $\delta \neq 0$ and $\Gamma = 0$.

This theorem can be proved by systematically checking all possibilities. Starting from (5), condition (4) can be integrated to give an expression for $T$ and a number of relations between the Lie algebra generators. Then applying the Jacobi identity to find new relations in all cases except for the four cases mentioned above, the prolongation structure is found to be finite-dimensional. In fact, apart from the cases (i)–(iv), it is always a subspace of the linear space $\langle x_1, z_1, \overline{z}_1, z_2, \overline{z}_2, x_2, x_3, y_1 \rangle$ where $y_1 = [z_1, \overline{z}_1]$.

Case (i) and (ii) are the derived NLS equation of type I and II. These are very similar, and they will be treated together in section 3. Case (iii) is the Hirota equation. It will be treated in section 4. For case (iv) Sasa and Satsuma recently found a soliton solution. Here, the resulting Lie algebra turns out to be more complex than in the previous cases. This case will be treated in section 5.

For each of these four cases, we will give the defining relations, the expression for $T$, the structure of the infinite-dimensional Lie algebra and the related differential equations. As shown in [11], the prolongation method can be used to find (auto-) Bäcklund transforms. This will be the subject of future work.

## 3. Cases (i) and (ii): the derived NLS equation

### 3.1. Defining relations

Cases (i) and (ii) are very similar, and can be treated together. The parameters are given by $\varepsilon(\beta_1, \beta_2, \beta_3) = (0, \delta, \delta_3)$, $\delta$ real and $\Gamma = 0$, with (i) $\delta_3 = 0$ or (ii) $\delta_3 = -\delta$.

In expression (5), $z_2 = 0$ (and therefore $\overline{z}_2 = 0$). With new generators defined by

$$z_3 = [x_1, z_1] \qquad \text{and} \qquad y_1 = [z_1, \overline{z}_1] \tag{6}$$

the defining relations are given by

$$\begin{gathered}
[x_1, x_2] = 0 \qquad [x_1, y_1] - [z_1, \overline{z}_3] + [\overline{z}_1, z_3] = 0 \qquad [y_1, z_1] + 2z_1 - 2i\delta z_3 = 0 \\
[x_1, x_3] = 0 \qquad [x_2, z_1] - i\delta_3 z_1 = 0 \qquad [x_1, z_3] - 2i[x_3, z_1] = 0 \\
[z_1, z_3] = 0 \qquad \text{and} \qquad [x_2, x_3] + i([z_1, \overline{z}_3] - [\overline{z}_1, z_3]) = 0\,.
\end{gathered} \tag{7}$$

Note that complex conjugated relations like $[\overline{z}_1, \overline{z}_3] = 0$ are defining relations too.

The vector field $T$ of equation (3) is given by

$$\begin{aligned}
2T = z_1\left[\mathrm{i}\psi_x - (2\delta + \delta_3)\psi^2\overline{\psi}\right] &+ \overline{z}_1\left[-\mathrm{i}\overline{\psi}_x - (2\delta + \delta_3)\psi\overline{\psi}^2\right] \\
&+ x_2[-\mathrm{i}\psi\overline{\psi}_x + \mathrm{i}\psi_x\overline{\psi} - (3\delta + 2\delta_3)\psi^2\overline{\psi}^2] \\
&+ 2x_3 - \mathrm{i}z_3\psi + \mathrm{i}\overline{z}_3\overline{\psi} + \mathrm{i}y_1\psi\overline{\psi}\,.
\end{aligned} \tag{8}$$

### 3.2. Structure of the Lie algebra

Denote $W = \mathbb{C}[\lambda^2]\langle(1 + \mathrm{i}\lambda) \otimes e, (1 - \mathrm{i}\lambda) \otimes f, 1 \otimes h\rangle \subset \mathbb{C}[\lambda] \otimes A_1$, where $e$, $f$ and $h$ form the basis of $A_1 = sl(2)$, with standard relations $[e, f] = h$, $[e, h] = -2e$ and $[f, h] = 2f$. $W$ is the sub-algebra of $\mathbb{C}[\lambda] \otimes A_1$ generated by $\lambda^{2i} \otimes h$, $\lambda^{2i}(1 + \mathrm{i}\lambda) \otimes e$ and $\lambda^{2i}(1 - \mathrm{i}\lambda) \otimes f$.

*Theorem 2.* The algebra $E_{01x}$ with generators $x_1$, $z_1$, $\overline{z}_1$, $x_2$, $x_3$ and $z_3$, $\overline{z}_3$, $y_1$ and defining relations (6) and (7) is isomorphic to $W \oplus H_{01x}$. Here $H_{01x} = \langle c_2, c_3\rangle$ is the centre.

First we need a lemma for the defining relations of $W$.

*Lemma 1.* The algebra $W$ is isomorphic to the Lie algebra with generators $e_0$, $f_0$ and $h_0$ and defining relations

$$[e_0, h_0] = -2e_0 \qquad [f_0, h_0] = 2f_0 \qquad \text{and} \qquad (\mathrm{ad}\ f_0)^3 e_0 = (\mathrm{ad}\ e_0)^3 f_0 = 0 \tag{9}$$

via the isomorphism

$$e_0 \mapsto (1 + \mathrm{i}\lambda) \otimes e \qquad f_0 \mapsto (1 - \mathrm{i}\lambda) \otimes f \qquad \text{and} \qquad h_0 \mapsto 1 \otimes h\,. \tag{10}$$

*Proof.* We start by defining a basis of $W$. Define

$$h_{i+1} \text{ isomorphic to } \lambda^{2(i+1)} \otimes h \text{ by } [e_i, f_0] = h_i + h_{i+1}$$
$$e_i \text{ isomorphic to } \lambda^{2i}(1 + \mathrm{i}\lambda) \otimes e \text{ by } [e_0, h_i] = -2e_i$$
$$f_i \text{ isomorphic to } \lambda^{2i}(1 - \mathrm{i}\lambda) \otimes f \text{ by } [f_0, h_i] = 2f_i\,.$$

For the mapping $h_i \mapsto \lambda^{2i} \otimes h$ etc. to be an isomorphism, the following equations ought to be satisfied.

$$1^{ij} : [e_i, e_j] = 0 \qquad 2^{ij} : [e_i, f_j] = h_{i+j} + h_{i+j+1} \qquad 3^{ij} : [e_i, h_j] = -2e_{i+j}$$
$$4^{ij} : [f_i, f_j] = 0 \qquad 5^{ij} : [f_i, h_j] = 2f_{i+j} \qquad 6^{ij} : [h_i, h_j] = 0\,. \tag{11}$$

For conciseness, 'The Jacobi identity applied to $x$, $y$ and $z$ yields, given the already proved statements $p$, $q$,…, that' is written down as '$[x,y,z], p, q, \ldots \Rightarrow$'.

For $i = j = 0$, statements $1^{00}$, $4^{00}$ and $6^{00}$ are trivial, $3^{00}$ and $5^{00}$ are given, and $2^{00}$ is true by definition.

For $i + j = 1$, statements $2^{01}$, $3^{01}$, $5^{01}$ are true by definition. Statement $1^{01}$ (and $1^{10}$) follows from $0 = (\text{ad } e_0)^3 f_0 = (\text{ad } e_0)^2(h_0 + h_1) = [e_0, -2e_0 - 2e_1] = -2[e_0, e_1]$. Similarly, statement $4^{01}$ (and $4^{10}$) follow from $(\text{ad } f_0)^3 e_0 = 0$. Statement $6^{01}$ (and $6^{10}$) follows from $[e_0, f_0, h_0], 2^{00}, 3^{00}, 5^{00} \Rightarrow [h_0, h_1] = 0$.

Statement $3^{10}$ follows from $[e_0, h_0, h_1], 6^{01}, 3^{01}, 3^{00} \Rightarrow [h_0, e_1] = 2e_1$ and similarly statement $5^{10}$ follows from $[f_0, h_0, h_1], 6^{01}, 5^{01}, 5^{00} \Rightarrow [h_0, f_1] = -2f_1$.

Now suppose $1^{ij}$, $3^{ij}$, $4^{ij}$, $5^{ij}$ and $6^{ij}$ are known for all $i, j$ with $i + j \leqslant n$ and $2^{ij}$ is known for all $i, j$ with $i + j \leqslant n - 1$, which we know to be true for $n = 1$. Then we can prove the following.

*$2^{ij}$ and $6^{ij}$.* Let $i + j = n - 1$. Then $[e_i, f_j, h_1], 5^{j1}, 3^{i1}, 2^{ij}, 6^{1,i+j} \Rightarrow [e_i, f_{j+1}] = [e_{i+1}, f_j] - \frac{1}{2}[h_1, h_n]$. Because $[e_n, f_0] = h_n + h_{n+1}$ it follows by induction on $j$, that $[e_{n-j}, f_j] = h_n + h_{n+1} - \frac{1}{2}j[h_1, h_n]$.

Now let $i + k = n$ and $k \geqslant 1$. Then, using the above, $[e_i, f_0, h_k], 5^{0k}, 3^{ik}, 2^{i0}, 6^{ik} \Rightarrow [h_k, h_{n+1-k}] = k[h_1, h_n]$ (for all $k \geqslant 1$). Take $k = n$: $[h_n, h_1] = -[h_1, h_n] = n[h_1, h_n]$, so $[h_1, h_n] = 0$.

Now $2^{ij}$ has been proved for all $i, j$ for which $i + j = n$, and $6^{ij}$ for all $i, j$ for which $i + j = n + 1$, except for $6^{0,n+1}$ (and $6^{n+1,0}$). For that case, $[e_n, f_0, h_0], 5^{00}$, $3^{n0}, 2^{n0}$, $6^{0n} \Rightarrow [h_0, h_{n+1}] = 0$.

*$3^{ij}$.* $3^{i,j+1}$ with $i + j = n$ can be proved by $[e_0, e_i, f_j], 2^{0j}, 2^{ij}, 1^{0i}, 3^{ij}, 3^{0n}, 3^{0,n+1} \Rightarrow [e_i, h_{j+1}] = -2e_{n+1}$ ($3^{0,n+1}$ is the definition of $e_{n+1}$). Now $[e_1, h_0, h_n], 3^{10}, 3^{1n}$, $6^{n0} \Rightarrow [e_{n+1}, h_0] = -2e_{n+1}$, which proves the final case $3^{n+1,0}$

*$5^{ij}$.* Likewise, $5^{i,j+1}$ with $i + j = n$ can be proved from $[f_0, e_i, f_j], 4^{0j}, 2^{ij}, 2^{i0}, 5^{ij}$, $5^{n0}, 5^{0,n+1} \Rightarrow [f_i, h_{j+1}] = 2f_{n+1}$. Now $5^{n+1,0}$ can be proved from $[f_1, h_0, h_n], 5^{10}, 5^{1n}$ and $6^{n0}$.

*$1^{ij}$.* For $j, k \geqslant 1$ and $j + k = n + 1$ (so $j, k \leqslant n$), $[e_0, e_k, h_j], 3^{kj}, 1^{0k}, 3^{0j} \Rightarrow [e_0, e_{k+j}] = [e_j, e_k]$; the same holds with $j$ and $k$ interchanged, so $[e_0, e_{k+j}] = [e_j, e_k] = -[e_k, e_j] = -[e_0, e_{j+k}]$, and thus $[e_0, e_{k+j}] = [e_j, e_k] = 0$. Now $1^{jk}$ has been proved for all $j, k$ with $j + k = n + 1$, except for $1^{1n}$ and $1^{n1}$; this case can be dealt with by $[e_1, e_{n-1}, h_1], 3^{n-1,1}, 3^{11}, 1^{1,n-1} \Rightarrow [e_1, e_n] = 0$.

*4ij.* The proof of $4^{ij}$ is similar; use $[f_0, f_k, h_j], 5^{kj}, 4^{0k}, 5^{0j} \Rightarrow \ldots$ and $[f_1, f_{n-1}, h_1]$, $5^{n-1,1}, 5^{11}, 4^{1,n-1} \Rightarrow \ldots$ respectively.

This finishes the proof of the lemma, and now we can prove the theorem. In order to give the isomorphism we let $\lambda = \delta\mu$. Denote the free Lie algebra on generators $x, y, \ldots$ by $L(x, y, \ldots)$. Consider the Lie algebra morphism $\phi_{01x}$ : $L(x_1, z_1, \overline{z}_1, x_2, x_3, z_3, \overline{z}_3, y_1) \mapsto W \oplus H_{01x}$ given by

$$\begin{aligned}
&x_1 \mapsto \frac{i}{2}\delta\mu^2 \otimes h && z_1 \mapsto (1 + i\delta\mu) \otimes e && z_3 \mapsto i\delta\mu^2(1 + i\delta\mu) \otimes e \\
&x_2 \mapsto \frac{i}{2}\delta_3 \otimes h \oplus c_2 && \overline{z}_1 \mapsto -(1 - i\delta\mu) \otimes f && \overline{z}_3 \mapsto i\delta\mu^2(1 - i\delta\mu) \otimes f \\
&x_3 \mapsto \frac{i}{4}\delta^2\mu^4 \otimes h \oplus c_3 && y_1 \mapsto -(1 + \delta^2\mu^2) \otimes h\,. &&
\end{aligned} \tag{12}$$

$\phi_{01x}$ preserves the defining relations of $E_{01x}$, (6) and (7). Therefore there exists a Lie algebra morphism $\phi'_{01x} : E_{01x} \mapsto W \oplus H_{01x}$.

For the inverse morphism, consider the morphism $\chi_{01x} : L(e_0, f_0, h_0, c_1, c_2) \mapsto E_{01x}$ defined by

$$\begin{aligned}
&e_0 \mapsto z_1 && c_2 \mapsto x_2 + \frac{i}{2}\delta_3 y_1 + \delta\delta_3 x_1 \\
&f_0 \mapsto -\overline{z}_1 && c_3 \mapsto x_3 - \frac{1}{4\delta}x_4 + \frac{1}{2\delta}x_1 \\
&h_0 \mapsto -y_1 + 2i\delta x_1\,. &&
\end{aligned} \tag{13}$$

As can be checked easily from table 1, $\chi_{01x}$ leaves relations (9) invariant, and $c_2$ and $c_3$ are mapped to central elements, hence there is a Lie algebra morphism $\chi'_{01x}$ : $W \oplus H_{01x} \mapsto E_{01x}$ as well. $\chi'_{01x}$ and $\phi'_{01x}$ are each other's inverse, so $\phi'_{01x}$ is a Lie algebra isomorphism.

**Table 1.** The Lie product table for $\varepsilon\,(\beta_1, \beta_2, \beta_3) = (0, \delta, \delta_3)$ with $\delta_3 = 0$ or $\delta_3 = -\delta$. All products of the form $[x_i, x_j]$, $[x_i, y_j]$, $[y_i, y_j]$, $[z_i, z_j]$ and $[\overline{z}_i, \overline{z}_j]$ are zero.

| | $z_1$ | $z_3$ |
|---|---|---|
| $x_1$ | $z_3$ | $z_4$ |
| $x_2$ | $i\delta_3 z_1$ | $i\delta_3 z_3$ |
| $x_3$ | $-\frac{i}{2}z_4$ | $-\frac{i}{2}z_5$ |
| $x_4$ | $-2i\delta z_4 + 2z_3$ | $-2i\delta z_5 + 2z_4$ |
| $y_1$ | $2i\delta z_3 - 2z_1$ | $2i\delta z_4 - 2z_3$ |
| $\overline{z}_1$ | $-y_1$ | $x_4$ |
| $\overline{z}_3$ | $-x_4$ | $y_2$ |

### *3.3. A realization*

Using the nonlinear representation of the algebra $sl(2)$

$$e = -y^2\partial_y \qquad f = \partial_y \qquad h = 2y\partial_y \tag{14}$$

the following equations result.

$$\psi_t = \frac{i}{2}\psi_{xx} + i\psi^2\overline{\psi} - (2\delta + \delta_3)\psi\overline{\psi}\psi_x - (\delta + \delta_3)\psi^2\overline{\psi}_x \tag{15}$$

$$y_x = -(1 + \delta i\mu)y^2\psi + y(i\delta\mu^2 + i\delta_3\psi\overline{\psi}) + (-1 + i\delta\mu)\overline{\psi} \tag{16}$$

$$\begin{aligned} y_t = {} & \tfrac{1}{2}y^2(1 + i\delta\mu)[-i\psi_x + (2\delta + \delta_3)\psi^2\overline{\psi} - \delta\mu^2\psi] \\ & + \tfrac{1}{2}y\big[-\delta_3\psi_x\overline{\psi} + \delta_3\overline{\psi}_x\psi + i\delta^2\mu^4 - 2i(1 + \delta^2\mu^2)\psi\overline{\psi} - i\delta_3(3\delta + 2\delta_3)\psi^2\overline{\psi}^2\big] \\ & + \tfrac{1}{2}(1 - i\delta\mu)\big[-i\overline{\psi}_x + (2\delta + \delta_3)\psi\overline{\psi}^2 - \delta\mu^2\overline{\psi}\big] \end{aligned} \tag{17}$$

and for the radical, using the representation $c_2 = \partial_{w_2}$ and $c_3 = \partial_{w_3}$,

$$w_{2x} = \psi\overline{\psi} \tag{18}$$

$$w_{2t} = \tfrac{1}{2}\big[i\overline{\psi}\psi_x - i\psi\overline{\psi}_x - (3\delta + 2\delta_3)\psi^2\overline{\psi}^2\big] \tag{19}$$

and $w_3 = t$. Only $(\partial/\partial t)w_{2x} = (\partial/\partial x)w_{2t}$ gives a conservation law.

## 4. Case (iii): the Hirota equation

### *4.1. Defining relations*

For the Hirota equation, the parameters are $\varepsilon(\beta_1, \beta_2, \beta_3) = \delta(1, 6, -6)$ ($\delta$ real) and $\Gamma = 0$. As in the previous section, $z_2 = 0$. With new generators defined by

$$y_1 = [z_1, \overline{z}_1] \qquad z_3 = [x_1, z_1] \qquad \text{and} \qquad z_4 = [x_1, z_3] \tag{20}$$

the defining relations are given by

$$\begin{aligned} &[x_1, x_2] = 0 \qquad [z_1, \overline{z}_3] - [\overline{z}_1, z_3] + 2i[x_1, y_1] = 0 \qquad [z_1, z_3] = 0 \\ &[x_1, x_3] = 0 \qquad 2[x_2, x_3] - 3\delta([z_1, \overline{z}_4] + [\overline{z}_1, z_4]) = 0 \qquad [y_1, z_1] + 2z_1 = 0 \\ &[x_2, z_1] = 0 \qquad [x_3, z_1] + \delta[x_1, z_4] + \frac{i}{2}z_4 = 0 . \end{aligned} \tag{21}$$

The vector field $T$ of equation (3) is given by

$$\begin{aligned} T = {} & z_1\left(-2\delta\psi^2\overline{\psi} - \delta\psi_{xx} + \frac{i}{2}\psi_x\right) + \overline{z}_1\left(-2\delta\psi\overline{\psi}^2 - \delta\overline{\psi}_{xx} - \frac{i}{2}\overline{\psi}_x\right) \\ & + x_2\left(-3\delta\psi^2\overline{\psi}^2 - \delta\psi\overline{\psi}_{xx} + \delta\psi_x\overline{\psi}_x - \delta\psi_{xx}\overline{\psi} - \frac{i}{2}\psi\overline{\psi}_x + \frac{i}{2}\psi_x\overline{\psi}\right) \\ & + x_3 + z_3\left(\delta\psi_x - \frac{i}{2}\psi\right) + \overline{z}_3\left(\delta\overline{\psi}_x + \frac{i}{2}\overline{\psi}\right) \\ & + y_1\left(\delta\psi\overline{\psi}_x - \delta\psi_x\overline{\psi} + \frac{i}{2}\psi\overline{\psi}\right) - \delta z_4\psi - \delta\overline{z}_4\overline{\psi} - \delta x_4\psi\overline{\psi} . \end{aligned} \tag{22}$$

### 4.2. Structure of the Lie algebra

*Theorem 3.* The algebra $E_{166}$ with generators $x_1$, $z_1$, $\overline{z}_1$, $x_2$, $x_3$ and $y_1$, $z_3$, $\overline{z}_3$, $z_4$, $\overline{z}_4$ and defining relations (20) and (21) is isomorphic to $\mathbb{C}[\lambda] \otimes A_1 \oplus H_{166}$, where $H_{166} = \langle c_1, c_2, c_3 \rangle$ is the centre.

The proof of the following lemma can be found in [7].

*Lemma 2.* $\mathbb{C}[\lambda] \otimes A_1$ is isomorphic to the Lie algebra with generators $e_0$, $f_0$, $h_0$, $e_1$ and defining relations

$$\begin{aligned} &[e_0, f_0] = h_0 \qquad && [h_0, e_1] = -2e_1 \qquad && (\mathrm{ad}\ e_0)^3 e_1 = 0 \\ &[e_0, h_0] = -2e_0 && [f_0, e_1] = 0 && (\mathrm{ad}\ e_1)^3 e_0 = 0 \\ &[f_0, h_0] = 2f_0 \end{aligned} \tag{23}$$

via the isomorphism defined by

$$e_0 \mapsto 1 \otimes e \qquad f_0 \mapsto 1 \otimes f \qquad h_0 \mapsto 1 \otimes h \qquad \text{and} \qquad e_1 \mapsto \lambda \otimes f\,. \tag{24}$$

Like in the previous section, the Lie algebra morphism

$$\phi_{166} : L(x_1, z_1, \overline{z}_1, x_2, x_3, y_1, z_3, \overline{z}_3, z_4, \overline{z}_4) \mapsto \mathbb{C}[\lambda] \otimes A_1 \oplus H_{166}$$

given by

$$\begin{aligned} &x_1 \mapsto \mathrm{i}\lambda \otimes h \oplus c_1 \qquad z_1 \mapsto 1 \otimes e \qquad \overline{z}_1 \mapsto -1 \otimes f \\ &x_2 \mapsto c_2 \qquad z_3 \mapsto 2\mathrm{i}\lambda \otimes e \qquad \overline{z}_3 \mapsto 2\mathrm{i}\lambda \otimes f \\ &x_3 \mapsto (4\mathrm{i}\delta\lambda^3 + \mathrm{i}\lambda^2) \otimes h \oplus c_3 \qquad z_4 \mapsto -4\lambda^2 \otimes e \qquad \overline{z}_4 \mapsto 4\lambda^2 \otimes f \\ &y_1 \mapsto -1 \otimes h \end{aligned} \tag{25}$$

leaves the relations (20) and (21) invariant, so there is a morphism $\phi'_{166} : E_{166} \mapsto \mathbb{C}[\lambda] \otimes A_1 \oplus H_{166}$. For the inverse mapping, consider the morphism $\chi_{166}$ : $L(e_0, f_0, h_0, e_1, c_1, c_2, c_3) \mapsto E_{166}$ defined by

$$\begin{aligned} &e_0 \mapsto z_1 \qquad && e_1 \mapsto -\frac{\mathrm{i}}{2}\overline{z}_3 \qquad && c_1 \mapsto x_1 - \frac{1}{2}x_4 \\ &f_0 \mapsto -\overline{z}_1 && && c_2 \mapsto x_2 \\ &h_0 \mapsto -y_1 && && c_3 \mapsto x_3 - \frac{\mathrm{i}}{4}y_2 + \frac{1}{2}\delta x_5\,. \end{aligned} \tag{26}$$

The new generators introduced here are given by $x_4 = [z_1, \overline{z}_3]$, $y_2 = [z_1, \overline{z}_4]$, $z_5 = [x_1, z_4]$ and $x_5 = [z_1, \overline{z}_5]$. As can be seen from table 2, $\chi_{166}$ leaves relations (23) invariant, and $c_1$, $c_2$ and $c_3$ are mapped to central elements. So there is a morphism $\chi'_{166} : \mathbb{C}[\lambda] \otimes A_1 \oplus H_{166} \mapsto E_{166}$, which can be checked to be the inverse of $\phi'_{166}$. So $\phi'_{166}$ is an isomorphism, which concludes the proof.

**Table 2.** The Lie product table for $\varepsilon(\beta_1,\beta_2,\beta_3) = \delta(1,6,-6)$. All products of the form $[x_i,x_j]$, $[x_i,y_j]$, $[y_i,y_j]$, $[z_i,z_j]$ and $[\overline{z}_i,\overline{z}_j]$ are zero.

| | $z_1$ | $z_3$ | $z_4$ | $z_5$ |
|---|---|---|---|---|
| $x_1$ | $z_3$ | $z_4$ | $z_5$ | $z_6$ |
| $x_2$ | 0 | 0 | 0 | 0 |
| $x_3$ | $-\frac{i}{2}z_4-\delta z_5$ | $-\frac{i}{2}z_5-\delta z_6$ | $-\frac{i}{2}z_6-\delta z_7$ | $-\frac{i}{2}z_7-\delta z_8$ |
| $x_4$ | $2z_3$ | $2z_4$ | $2z_5$ | $2z_6$ |
| $x_5$ | $2z_5$ | $2z_6$ | $2z_7$ | $2z_8$ |
| $y_1$ | $-2z_1$ | $-2z_3$ | $-2z_4$ | $-2z_5$ |
| $y_2$ | $-2z_4$ | $-2z_5$ | $-2z_6$ | $-2z_7$ |
| $\overline{z}_1$ | $-y_1$ | $x_4$ | $-y_2$ | $x_5$ |
| $\overline{z}_3$ | $-x_4$ | $y_2$ | $-x_5$ | $y_3$ |
| $\overline{z}_4$ | $-y_2$ | $x_5$ | $-y_3$ | $x_6$ |
| $\overline{z}_5$ | $-x_5$ | $y_3$ | $-x_6$ | $y_4$ |

### *4.3. A representation*

With the same representation of $sl(2)$ as in section 3.3, the following system of equations result.

$$\psi_t = \frac{i}{2}\psi_{xx} + i\psi^2\overline{\psi} - \delta\psi_{xxx} - 6\delta\psi\overline{\psi}\psi_x \tag{27}$$

$$y_x = -y^2\psi + 2i\lambda y - \overline{\psi} \tag{28}$$

$$\begin{aligned} y_t = y^2 &\left[\delta\psi_{xx} - \frac{i}{2}(1+4\delta\lambda)\psi_x - \lambda(1+4\delta\lambda)\psi + 2\delta\psi^2\overline{\psi}\right] \\ &+ y\left[2\delta\psi_x\overline{\psi} - 2\delta\overline{\psi}_x\psi + 2i\lambda^2(1+4\delta\lambda) - i\psi\overline{\psi}(1+4\delta\lambda)\right] \\ &+ \left[\delta\overline{\psi}_{xx} + \frac{i}{2}(1+4\delta\lambda)\overline{\psi}_x - \lambda(1+4\delta\lambda)\overline{\psi} + 2\delta\psi\overline{\psi}^2\right]. \end{aligned} \tag{29}$$

Representing $c_1$ by $\partial_{w_1}$, $c_2$ by $\partial_{w_2}$ and $c_3$ by $\partial_{w_3}$, the radical is given by $w_1 = x$, $w_3 = t$, and

$$w_{2x} = \psi\overline{\psi} \tag{30}$$

$$w_{2t} = \frac{i}{2}(\psi_x\overline{\psi} - \overline{\psi}_x\psi) - \delta\psi_{xx}\overline{\psi} - \delta\overline{\psi}_{xx}\psi + \delta\psi_x\overline{\psi}_x - 3\delta\psi^2\overline{\psi}^2. \tag{31}$$

Again, only $w_2$ gives a conservation law.

## 5. The final case: case (iv)

### *5.1. Defining relations*

The last case, $\varepsilon(\beta_1,\beta_2,\beta_3) = \delta(1,6,-3)$, leads to a more complex prolongation structure. Not only the radical, but also the loop-algebra part of the prolongation structure gets more involved. Whereas in the previous cases, the prolongation algebra was a sub-algebra of the Kac–Moody algebra $A_1^{(1)}$, in case (iv) it is a sub-algebra of the *twisted* algebra $A_2^{(2)}$. This implies that a nonlinear representation of the regular part has to be at least two-dimensional.

The generators $z_2$ and $\overline{z}_2$ do not have to be zero any more (they are still only part of the radical, though). Introducing new generators

$$\begin{aligned} &x_4 = [\overline{z}_1, z_3] \qquad y_1 = [z_1, \overline{z}_1] \\ &z_3 = [x_1, z_1] \qquad z_4 = [x_1, z_3] \qquad z_5 = [x_1, z_4] \end{aligned} \tag{32}$$

the defining relations are given by

$$\begin{aligned} &[x_1, x_2] = 0 \qquad [z_1, z_2] = 0 \qquad [y_1, z_1] + z_1 = 0 \\ &[x_1, x_3] = 0 \qquad [z_1, \overline{z}_2] = 0 \qquad [x_3, z_1] + \delta[x_1, z_4] + \frac{\mathrm{i}}{2}[x_1, z_3] = 0 \\ &[x_1, y_1] = 0 \qquad [x_2, z_2] = 0 \qquad [z_1, \overline{z}_5] + 2\overline{z}_3 - \frac{\mathrm{i}}{3\delta}\overline{z}_1 \\ &[x_2, z_1] = 0 \qquad [z_2, \overline{z}_2] = 0 \qquad [x_2, x_3] + [z_1, \overline{z}_4] + [\overline{z}_1, z_4] = 0 \\ &[z_1, z_5] = 0 \qquad [x_1, z_2] + \frac{\mathrm{i}}{3\delta} z_2 = 0 \qquad 2[x_3, z_2] - \frac{\mathrm{i}}{27\delta^2} z_2 + 3\delta[z_1, z_4] + \mathrm{i} z_5 = 0\,. \end{aligned} \tag{33}$$

$[x_1, y_1] = 0$ is equivalent to $x_4$ being real.

The vector field $T$ of equation (3) is given by

$$\begin{aligned} T = {}& z_1\left(-4\delta\psi^2\overline{\psi} - \delta\psi_{xx} + \frac{\mathrm{i}}{2}\psi_x\right) + \overline{z}_1\left(-4\delta\psi\overline{\psi}^2 - \delta\overline{\psi}_{xx} - \frac{\mathrm{i}}{2}\overline{\psi}_x\right) \\ & + z_2\left(-6\delta\psi^3\overline{\psi} - 2\delta\psi\psi_{xx} + \delta\psi_x^2 - \frac{1}{18\delta}\psi^2 + \frac{\mathrm{i}}{3}\psi\psi_x\right) \\ & + \overline{z}_2\left(-6\delta\psi\overline{\psi}^3 - 2\delta\overline{\psi\psi}_{xx} + \delta\overline{\psi}_x^2 - \frac{1}{18\delta}\overline{\psi}^2 - \frac{\mathrm{i}}{3}\overline{\psi\psi}_x\right) \\ & + x_2\left(-6\delta\psi^2\overline{\psi}^2 - \delta\psi\overline{\psi}_{xx} + \delta\psi_x\overline{\psi}_x - \delta\psi_{xx}\overline{\psi} - \frac{\mathrm{i}}{2}\psi\overline{\psi}_x + \frac{\mathrm{i}}{2}\psi_x\overline{\psi}\right) \\ & + x_3 + z_3\left(\delta\psi_x - \frac{\mathrm{i}}{2}\psi\right) + \overline{z}_3\left(\delta\overline{\psi}_x + \frac{\mathrm{i}}{2}\overline{\psi}\right) + y_1\left(\delta\psi\overline{\psi}_x - \delta\psi_x\overline{\psi} + \frac{\mathrm{i}}{2}\psi\overline{\psi}\right) \\ & - \delta z_4\psi - \delta\overline{z}_4\overline{\psi} - \frac{1}{2}\delta z_5\psi^2 - \delta x_4\psi\overline{\psi} - \frac{1}{2}\delta\overline{z}_5\overline{\psi}^2\,. \end{aligned} \tag{34}$$

### *5.2. Structure of the Lie algebra*

As mentioned above, in case (iv) the prolongation algebra turns out to be a sub-algebra of the *twisted* Kac–Moody algebra $A_2^{(2)}$. A realization of this algebra can be found in Kac [3]. More specifically, if we write $A_2 = sl(3) = A_{2\overline{0}} \oplus A_{2\overline{1}}$, with $A_{2\overline{0}} = \langle e, f, h\rangle$ and $A_{2\overline{1}} = \langle v_{-4}, v_{-2}, v_0, v_2, v_4\rangle$, with multiplication table given by table 3, $A_2^{(2)}$ modulo its centre is isomorphic to the algebra $\bigoplus_{k=-\infty}^{\infty} \lambda^k \otimes A_{2\overline{k}} \subset \mathbb{C}[\lambda, \lambda^{-1}] \otimes A_2$, with $\overline{k} = k \bmod 2$.

Let $K = \bigoplus_{k=1}^{\infty} \lambda^k \otimes A_{2\overline{k}} \subset A_2^{(2)}$. We find

**Table 3.** Mmultiplication table for the $A_2$.

| | $e$ | $f$ | $h$ | $v_{-4}$ | $v_{-2}$ | $v_0$ | $v_2$ | $v_4$ |
|---|---|---|---|---|---|---|---|---|
| $e$ | 0 | $h$ | $-2e$ | $v_{-2}$ | $2v_0$ | $3v_2$ | $4v_4$ | 0 |
| $f$ | | 0 | $2f$ | 0 | $4v_{-4}$ | $3v_{-2}$ | $2v_0$ | $v_2$ |
| $h$ | | | 0 | $-4v_{-4}$ | $-2v_{-2}$ | 0 | $2v_2$ | $4v_4$ |
| $v_{-4}$ | | | | 0 | 0 | 0 | $-2f$ | $-h$ |
| $v_{-2}$ | | | | | 0 | $6f$ | $2h$ | $2e$ |
| $v_0$ | | | | | | 0 | $-6e$ | 0 |
| $v_2$ | | | | | | | 0 | 0 |
| $v_4$ | | | | | | | | 0 |

*Theorem 4.* The Lie algebra $E_{163}$ with generators $x_1$, $z_1$, $\overline{z}_1$, $z_2$, $\overline{z}_2$, $x_2$, $x_3$, and $x_4$, $y_1$, $z_3$, $\overline{z}_3$, $z_4$, $\overline{z}_4$, $z_5$, $\overline{z}_5$ and defining relations (32) and (33) is isomorphic to $K \oplus H_{163}$, where $H_{163} = \langle c_1, c_2, c_3, d_1, d_2 \rangle$, $[H_{163}, K] = \{0\}$ and within $H_{163}$ all but the following commutators are zero.

$$[c_1, d_1] = -\frac{i}{3\delta} d_1 \qquad [c_3, d_1] = -\frac{i}{54\delta^2} d_1$$
$$[c_1, d_2] = -\frac{i}{3\delta} d_2 \qquad [c_3, d_2] = -\frac{i}{54\delta^2} d_2 . \tag{35}$$

Similar to what is done in [7] for the positive part of the $A_1^{(1)}$, we can prove

*Lemma 3.* $K$ is isomorphic to the algebra generated by $e_0$, $f_0$, $h_0$, $e_1$ and defining relations

$$[e_0, f_0] = h_0 \qquad [h_0, e_1] = -4e_1 \qquad (\text{ad } e_0)^5 e_1 = 0$$
$$[e_0, h_0] = -2e_0 \qquad [f_0, e_1] = 0 \qquad (\text{ad } e_1)^2 e_0 = 0 \tag{36}$$
$$[f_0, h_0] = 2f_0$$

via the isomorphism defined by

$$e_0 \mapsto 1 \otimes e \qquad f_0 \mapsto 1 \otimes f \qquad h_0 \mapsto 1 \otimes h \qquad \text{and} \qquad e_1 \mapsto \lambda \otimes v_{-4} . \tag{37}$$

Let $L$ be the free Lie algebra on the generators of $E_{163}$ mentioned in theorem 4. Then the morphism $\phi_{163} : L \mapsto K \oplus H_{01x}$ given by

$$x_1 \mapsto -\frac{i}{12\delta} \otimes h - \frac{1}{18\delta} \lambda \otimes v_0 \oplus c_1 \qquad x_2 \mapsto c_2$$
$$x_3 \mapsto \frac{i}{216\delta^2} \otimes h + \frac{1}{216\delta^2}(\lambda + \tfrac{2}{3}\lambda^3) \otimes v_0 \oplus c_3 \qquad x_4 \mapsto -\frac{i}{12\delta} \otimes h - \frac{1}{6\delta} \lambda \otimes v_0$$
$$x_5 \mapsto -\frac{i}{216\delta^2} \otimes h - \frac{1}{216\delta^2}(3\lambda + 2\lambda^3) \otimes v_0 \qquad y_1 \mapsto -\frac{1}{2} \otimes h$$
$$z_3 \mapsto -\frac{i}{6\delta\sqrt{2}} \otimes e + \frac{1}{6\delta\sqrt{2}} \lambda \otimes v_2 \qquad z_1 \mapsto \frac{1}{\sqrt{2}} \otimes e$$
$$\overline{z}_3 \mapsto -\frac{i}{6\delta\sqrt{2}} \otimes f - \frac{1}{6\delta\sqrt{2}} \lambda \otimes v_{-2} \qquad \overline{z}_1 \mapsto -\frac{1}{\sqrt{2}} \otimes f$$
$$z_4 \mapsto \frac{1}{36\delta^2\sqrt{2}}(-1 + 2\lambda^2) \otimes e - \frac{i}{18\delta^2\sqrt{2}} \lambda \otimes v_2 \qquad z_5 \mapsto \frac{1}{3\delta} \lambda \otimes v_4$$
$$\overline{z}_4 \mapsto \frac{1}{36\delta^2\sqrt{2}}(1 - 2\lambda^2) \otimes f - \frac{i}{18\delta^2\sqrt{2}} \lambda \otimes v_{-2} \qquad \overline{z}_5 \mapsto \frac{1}{3\delta} \lambda \otimes v_{-4} \tag{38}$$

preserves (32) and (33), so there is a morphism $\phi'_{163} : E_{01x} \mapsto K \oplus H_{01x}$.

And, as can be checked from table 4, the mapping

$$\chi_{163} : L(e_0, f_0, e_1, c_1, c_2, d_1, d_2, d_3) \mapsto E_{163}$$

defined by

$$\begin{aligned} & e_0 \mapsto \sqrt{2} z_1 && c_1 \mapsto x_1 - \frac{1}{3} x_4 - \frac{\mathrm{i}}{9\delta} y_1 \\ & f_0 \mapsto -\sqrt{2}\overline{z}_1 && c_2 \mapsto x_2 \\ & h_0 \mapsto -2y_1 && c_3 \mapsto x_3 - \frac{1}{3} x_5 - \frac{\mathrm{i}}{162\delta^2} y_1 \\ & e_1 \mapsto 3\delta \overline{z}_5 && d_1 \mapsto z_2 \\ & && d_2 \mapsto \overline{z}_2 \end{aligned} \tag{39}$$

leaves (36) and (35) invariant, and also maps all other commutators with elements of $H_{163}$ to zero. So there is a morphism $\chi'_{163} : K \oplus H_{01x} \mapsto E_{01x}$. Again, the morphisms $\chi'_{163}$ and $\phi'_{163}$ are each other's inverse.

**Table 4.** The Lie product table for $\varepsilon(\beta_1, \beta_2, \beta_3) = \delta(1, 6, -3)$. All products of the form $[x_i, x_j]$, $[x_i, y_j]$, $[y_i, y_j]$, are zero, but products of the form $[z_i, z_j]$ and $[\overline{z}_i, \overline{z}_j]$ need not be.

| | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ |
|---|---|---|---|---|---|
| $x_1$ | $z_3$ | $-\frac{\mathrm{i}}{3\delta} z_2$ | $z_4$ | $\frac{1}{\delta} z_6 - \frac{\mathrm{i}}{2\delta} z_4$ | $-\frac{\mathrm{i}}{3\delta} z_5$ |
| $x_2$ | 0 | 0 | 0 | 0 | 0 |
| $x_3$ | $-z_6$ | $\frac{\mathrm{i}}{54\delta^2} z_2$ | $z_8$ | $z_9$ | $\frac{\mathrm{i}}{54\delta^2} z_5$ |
| $x_4$ | $3z_3 + \frac{\mathrm{i}}{3\delta} z_1$ | 0 | $3z_4 + \frac{\mathrm{i}}{3\delta} z_3$ | $\frac{3}{\delta} z_6 - \frac{7\mathrm{i}}{6\delta} z_4$ | $-\frac{\mathrm{i}}{3\delta} z_5$ |
| $x_5$ | $3z_6 + \frac{\mathrm{i}}{54\delta^2} z_1$ | 0 | $-3z_8 + \frac{\mathrm{i}}{54\delta^2} z_3$ | $-3z_9 + \frac{\mathrm{i}}{54\delta^2} z_4$ | $-\frac{\mathrm{i}}{54\delta^2} z_5$ |
| $y_1$ | $-z_1$ | 0 | $-z_3$ | $-z_4$ | $-2z_5$ |
| $z_1$ | 0 | 0 | $z_5$ | $-\frac{\mathrm{i}}{3\delta} z_5$ | 0 |
| $z_2$ | | 0 | 0 | 0 | 0 |
| $z_3$ | | | 0 | $-\frac{1}{\delta} z_7 + \frac{1}{18\delta^2} z_5$ | 0 |
| $z_4$ | | | | 0 | 0 |
| $z_5$ | | | | | 0 |
| $\overline{z}_1$ | $-y_1$ | 0 | $x_4$ | $y_2$ | $-2z_3 - \frac{\mathrm{i}}{3\delta} z_1$ |
| $\overline{z}_2$ | | 0 | 0 | 0 | 0 |
| $\overline{z}_3$ | | | $-y_2$ | $\frac{\mathrm{i}}{2\delta} y_2 - \frac{1}{\delta} x_5$ | $-2z_4 - \frac{\mathrm{i}}{\delta} z_3 + \frac{1}{9\delta^2} z_1$ |
| $\overline{z}_4$ | | | | $-y_3$ | $-\frac{2}{\delta} z_6 - \frac{2\mathrm{i}}{3\delta} z_4 + \frac{4}{9\delta^2} z_3 + \frac{\mathrm{i}}{27\delta^3} z_1$ |
| $\overline{z}_5$ | | | | | $-4y_2 - \frac{4\mathrm{i}}{3\delta} x_4 - \frac{1}{9\delta^2} y_1$ |

### 5.3. *A representation*

Since the $A_2$ contains two commuting elements, a vector-field representation has to be at least two-dimensional. A two-dimensional representation is given by

$$
\begin{aligned}
e &= \sqrt{2}(\partial_y + y\partial_z) & v_{-4} &= -\sqrt{2}(yz\partial_y + z^2\partial z) \\
f &= \sqrt{2}((-y^2+z)\partial_y - yz\partial_z) & v_{-2} &= -2(y^2+z)\partial_y - 2yz\partial_z \\
h &= -2y\partial_y - 4z\partial_z & v_0 &= -3\sqrt{2}y\partial_y \\
& & v_2 &= -2\partial_y + 2y\partial_z \\
& & v_4 &= \sqrt{2}\partial_z\,.
\end{aligned} \tag{40}
$$

With this representation, and changing to $\mu = (\mathrm{i}/\sqrt{2})\lambda$, the prolongation becomes

$$\psi_t = \frac{\mathrm{i}}{2}\psi_{xx} + \mathrm{i}\psi^2\overline{\psi} - \delta\psi_{xxx} - 9\delta\psi_x\psi\overline{\psi} - 3\delta\overline{\psi}_x\psi^2 \tag{41}$$

$$y_x = \frac{1-\mu}{6\delta}\mathrm{i}y + \psi + \overline{\psi}y^2 - \overline{\psi}z \tag{42}$$

$$z_x = \frac{\mathrm{i}}{3\delta}z + \psi y + \overline{\psi}yz \tag{43}$$

$$
\begin{aligned}
y_t = {} & -\delta\psi_{xx} - \delta\psi_x\overline{\psi}y + \frac{2+\mu}{6}\mathrm{i}\psi_x - \delta\overline{\psi}_{xx}y^2 + \delta\overline{\psi}_{xx}z + \delta\psi\overline{\psi}_x y \\
& - \frac{2+\mu}{6}\mathrm{i}\overline{\psi}_x y^2 + \frac{2-\mu}{6}\mathrm{i}\overline{\psi}_x z - 4\delta\psi^2\overline{\psi} - 4\delta\psi\overline{\psi}^2 y^2 + 4\delta\psi\overline{\psi}^2 z \\
& + \frac{\mu^2+\mu-2}{36\delta}\psi + \frac{\mu^2+\mu-2}{36\delta}\overline{\psi}y^2 + \frac{-\mu^2+\mu+2}{36\delta}\overline{\psi}z \\
& - \frac{\mu^3-3\mu+2}{216\delta^2}\mathrm{i}y + \frac{2+3\mu}{6}\mathrm{i}\psi\overline{\psi}y - \frac{\mu}{6}\mathrm{i}\overline{\psi}^2 yz
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
z_t = {} & -\delta\psi_{xx}y - 2\delta\psi_x\overline{\psi}z + \frac{2-\mu}{6}\mathrm{i}\psi_x y - \delta\overline{\psi}_{xx}yz + 2\delta\overline{\psi}_x\psi z - \frac{2+\mu}{6}\mathrm{i}\overline{\psi}_x yz \\
& - 4\delta\psi^2\overline{\psi}y - 4\delta\psi\overline{\psi}^2 yz + \frac{\mu^2-\mu-2}{36\delta}\psi y + \frac{\mu^2+\mu-2}{36\delta}\overline{\psi}yz \\
& - \frac{\mathrm{i}}{54\delta^2}z + \frac{2}{3}\mathrm{i}\psi\overline{\psi}z + \frac{\mu}{6}\mathrm{i}\psi^2 - \frac{\mu}{6}\mathrm{i}\overline{\psi}^2 z^2\,.
\end{aligned} \tag{45}
$$

The radical $H_{163}$ can be represented by

$$
\begin{aligned}
& c_1 = \frac{\mathrm{i}}{3\delta}(w_1\partial_{w_1} - w_2\partial_{w_2}) \qquad c_2 = \partial_{w_3} \qquad c_3 = -\frac{\mathrm{i}}{54\delta^2}(w_1\partial_{w_1} - w_2\partial_{w_2}) + \partial_{w_4} \\
& d_1 = \partial_{w_1} \qquad d_2 = \partial_{w_2}\,.
\end{aligned} \tag{46}
$$

This representations leads to

$$w_{1x} = \frac{\mathrm{i}}{3\delta}w_1 + \psi^2 \tag{47}$$

$$w_{2x} = -\frac{\mathrm{i}}{3\delta}w_2 + \overline{\psi}^2 \tag{48}$$

$$w_{3x} = \psi\overline{\psi} \tag{49}$$

$$w_{1t} = -2\delta\psi_{xx}\psi + \delta\psi_x^2 + \frac{\mathrm{i}}{3}\psi_x\psi - 6\delta\psi^3\overline{\psi} - \frac{1}{18\delta}\psi^2 - \frac{\mathrm{i}}{54\delta^2}w_1 \tag{50}$$

$$w_{2t} = -2\delta\overline{\psi}_{xx}\overline{\psi} + \delta\overline{\psi}_x^2 - \frac{\mathrm{i}}{3}\overline{\psi}_x\overline{\psi} - 6\delta\psi\overline{\psi}^3 - \frac{1}{18\delta}\overline{\psi}^2 + \frac{\mathrm{i}}{54\delta^2}w_2 \tag{51}$$

$$w_{3t} = -\delta\psi_{xx}\overline{\psi} + \delta\psi_x\overline{\psi}_x + \frac{\mathrm{i}}{2}\psi_x\overline{\psi} - \delta\overline{\psi}_{xx}\psi - \frac{\mathrm{i}}{2}\overline{\psi}_x\psi - 6\delta\psi^2\overline{\psi}^2 \tag{52}$$

and $w_4 = t$.

## 6. Conclusions

There are exactly four different parameter combinations which lead to infinite-dimensional prolongation structures, of which two are isomorphic. This suggests that these are the only four cases, besides the original NLS equation, for which equation (2) is completely integrable.

The differential equations that follow from the prolongation structures may be used to find Bäcklund transforms (cf [11]). Using such Bäcklund transforms the soliton solutions already found for these cases may be rediscovered. This will be the subject of future work.

It is also possible to construct a Lax pair from the prolongation structure, namely by taking a matrix representation of the prolongation algebra. In the light of this, the occurrence of $A_2^{(2)}$ in the fourth case is quite interesting. Namely, a matrix representation of $A_2^{(2)}$ and hence the Lax pair, is based on the finite-dimensional algebra $sl(3)$. In this way the Lax pair of [9] may be derived and it explains why in [9] the Lax pair was found by considering three-dimensional matrices.

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